Arithmetic Series
Warm-Up

Algebra 2 CST: Alg II 23.0	**Current: Alg I 17.0**
1. What would the nth term be in the arithmetic series below? $3+7+11+15+19+...$ A. 49 B. $3+4n$ C. $2n+1$ D. $4n-1$	**2.** Given the function $f(x)=3x-2$, find $f(1)+f(2)+f(3)+f(4)+f(5)$.
Review: Alg II 23.0	**Other: Alg II 23.0**
3. Kwanzaa is an African American harvest festival celebrating the new year. There is a ritual performed involving the lighting of 7 candles. On the first night a candle is lit and then blown out. On the second night, a new candle and the candle from the previous night is lit and then blown out. For seven nights this pattern of lighting a new candle and relighting all the candles from the previous nights continues. Fill in the chart showing the number of lightings for each day.	**4.** In problem 3, what is the total number of lightings during this festival?

Night	lightings
1	
2	
3	
4	
5	
6	
7	

Algebra II 23.0 Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

Arithmetic Series

Over-Arching Question:

A skydiver who jumps from an airplane falls 16 feet in the first second, 48 feet in the second second, 80 feet in the third second, and so on. How many feet would he/she fall in 20 seconds, if air resistance is ignored?

Definition: An **arithmetic series** is the sum of the terms in an arithmetic sequence**.**

S_n is used to denote the sum of the first *n* terms in a series.

Example 1) Find the sum: $-5+-1+3+7+11+15$.

There are 6 terms in this series.

$$S_6 = -5+-1+3+7+11+15$$
$$= 30$$

Review: How do we know the sequence in example 1 is arithmetic?
(It has a common difference of 4)

Find the rule for the sequence:

$$a_n = a_1 + (n-1)d$$
$$a_n = -5 + (n-1)4$$
$$a_n = -5 + 4n - 4$$
$$a_n = 4n - 9$$

Summation Notation, or **Sigma Notation,** is also used to indicate the sum of terms in a series. Σ is the Greek letter sigma.

For **example 1**, summation notation could look like this:

Add terms 1 through 6 → $\sum_{n=1}^{6}(4n-9)$ ← Rule for the sequence

Notice using the rule for the sequence, we can generate the 6 terms we want to add:

$a_1 = 4(1)-9 = -5$ $\quad a_2 = 4(2)-9 = -1$ $\quad a_3 = 4(3)-9 = 3$ $\quad a_4 = 4(4)-9 = 7$ $\quad a_5 = 4(5)-9 = 11$ $\quad a_6 = 4(6)-9 = 15$

So we evaluate as follows:

$$\sum_{n=1}^{6}(4n-9) = [(4(1)-9] + [(4(2)-9] + [(4(3)-9] + [(4(4)-9] + [(4(5)-9] + [(4(6)-9]$$
$$= -5+-1+3+7+11+15$$
$$= 30$$

Example 2) Write sigma notation for the sum $-3+2+7+12+17+22+27$.

The sequence is arithmetic with $d=5$ and $a_1=-3$.
The rule is

$$a_n = a_1 + (n-1)d$$
$$a_n = -3 + (n-1)5$$
$$a_n = 5n - 8$$

We are adding 7 terms

Sigma Notation: $\sum_{n=1}^{7}(5n-8)$

Example 3) ***You Try*** Write sigma notation for the sum $10+3+-4+-11$.

The sequence is arithmetic with $d=-7$ and $a_1=10$.
The rule is

$$a_n = a_1 + (n-1)d$$
$$a_n = 10 + (n-1)(-7)$$
$$a_n = 10 - 7n + 7$$
$$a_n = -7n + 17$$

Sigma Notation: $\sum_{n=1}^{4}(-7n+17)$

Quick Write: What is the difference between an arithmetic sequence and an arithmetic series?

An arithmetic sequence is a sequence of numbers that have a common difference, while an arithmetic series is a **sum** of numbers in an arithmetic sequence.

Developing a formula for adding the terms in an arithmetic series:

Example 4) Add the first 10 odd integers.

One way to write the sum: $S = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$

You can also write this in reverse order: $S = 19 + 17 + 15 + 13 + 11 + 9 + 7 + 5 + 3 + 1$

Add the two equations: $2S = 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20 + 20$

Simplify:

$$2S = (10)20$$
$$\frac{2S}{2} = \frac{(10)20}{2}$$
$$S = \frac{200}{2}$$
$$S = 100$$

$$2S_n = n\,(a_1 + a_n)$$
$$\frac{2S_n}{2} = \frac{n\,(a_1 + a_n)}{2}$$
$$S_n = \frac{n\,(a_1 + a_n)}{2}$$

In general, we multiply the sum of the first and last terms times *n*, then divide by 2.

Arithmetic Series Formula: The sum S_n of the first *n* terms of an arithmetic sequence is

$$S_n = \frac{n\,(a_1 + a_n)}{2}$$

a_n refers to the last term in the group of numbers we are adding.
n is the position of that last term.

Example 5) Find the sum of the first 27 terms in the series $-14 - 8 - 2 - ... + 142$.

To use our formula, we need to know that the sequence is arithmetic. Then we need the first term a_1, the last term a_n, and the position *n* of the last term.

We can see that the sequence is arithmetic since there is a common difference of 6.
We already know the first term, so $a_1 = -14$.
Since we are finding the sum of the first 27 terms, $n = 27$.
Since the last term shown is 142, we assume it is in the 27th position, so a_{27} =142.

Use the formula to find the sum:

$$S_n = \frac{n(a_1 + a_n)}{2}$$
$$S_{27} = \frac{27(-14 + 142)}{2}$$
$$= \frac{27(128)}{2}$$
$$= 1728$$

Example 6) Find the sum $\sum_{k=3}^{7}(2k+5)$

Method 1 Find the terms in the series, then add.

Replace k with 3 and then with 4, 5, 6, and 7 to find the terms.

$$\sum_{k=3}^{7}(2k+5)$$
$$=[2(3)+5]+[2(4)+5]+[2(5)+5]+[2(6)+5]+[2(7)+5]$$
$$=11+13+15+17+19$$
$$=75$$

Method 2 Use the series formula

We can see that we will be adding 5 terms since the values of k go from 3 to 7, so $n=5$

Use the rule $2k+5$ to find the first term by replacing k with 3 (the first value indicated for k). We will call this term a_1:

$$a_1 = 2(3)+5$$
$$=11$$

Find the last term by replacing k with 7 (the last value indicated for k). We will call this term a_5:

$$a_5 = 2(7)+5$$
$$=19$$

$$S_n = \frac{n(a_1+a_n)}{2}$$
$$S_5 = \frac{5(11+19)}{2}$$
$$=75$$

Example 7) ***You Try*** Find the sum $\sum_{j=5}^{8}(-3j+2)$. Use both methods.

Method 1

Find the terms in the series, then add.

$$\sum_{j=5}^{8}(-3j+2)$$
$$=[-3(5)+2]+[-3(6)+2]+[-3(7)+2]+[-3(8)+2]$$
$$=(-13)+(-16)+(-19)+(-22)$$
$$=-70$$

Method 2

Using the series formula:

$$n=4$$

$$a_1 = -3(5)+2$$
$$=-13$$

$$a_4 = -3(8)+2$$
$$=-22$$

$$S_n = \frac{n(a_1+a_n)}{2}$$
$$S_4 = \frac{4(-13+-22)}{2}$$
$$=\frac{4(-35)}{2}$$
$$=-70$$

When the last term is unknown:

The arithmetic **sequence** formula can be used to find any term in an arithmetic sequence, so if you don't know the last term in the series of numbers you are adding, you can use the arithmetic sequence formula to find it.

Example 8) Find the sum of the first 25 terms in the series $16+10+4-2-...$

We know the sequence is arithmetic with $d=-6$ and $a_1=16$. We need to find a_{25}.
Use the arithmetic **sequence** formula:

$$\begin{aligned} a_n &= a_1+(n-1)d \\ a_{25} &= 16+(25-1)(-6) \\ &= 16+(24)(-6) \\ &= -128 \end{aligned}$$

Now use the arithmetic **series** formula:

$$\begin{aligned} S_n &= \frac{n(a_1+a_n)}{2} \\ S_{27} &= \frac{25(16+-128)}{2} \\ &= \frac{25(-112)}{2} \\ &= -1400 \end{aligned}$$

Example 9) ***You Try*** Find the sum of the first 40 terms in the series $100+92+84+76+...$

We know the sequence is arithmetic with $d=-8$ and $a_1=100$. We need to find a_{40}.
Use the arithmetic **sequence** formula:

$$\begin{aligned} a_n &= a_1+(n-1)d \\ a_{40} &= 100+(40-1)(-8) \\ &= 100+(39)(-8) \\ &= 100-312 \\ &= -212 \end{aligned}$$

Now use the arithmetic **series** formula:

$$\begin{aligned} S_n &= \frac{n(a_1+a_n)}{2} \\ S_{40} &= \frac{40(100+-212)}{2} \\ &= \frac{40(-112)}{2} \\ &= -2240 \end{aligned}$$

Revisiting the Over-Arching Question:

A skydiver who jumps from an airplane falls 16 feet in the first second, 48 feet in the second second, 80 feet in the third second, and so on. How many feet would he/she fall in 20 seconds, if air resistance is ignored?

We need to add the first twenty terms of the arithmetic series $16+48+80+...$

$$\begin{aligned} a_{20} &= 16+(20-1)(32) \\ &= 16+(19)(32) \\ &= 16+608 \\ &= 624 \end{aligned}$$

$$\begin{aligned} S_{20} &= \frac{20(16+624)}{2} \\ &= \frac{20(640)}{2} \\ &= 6400 \end{aligned}$$

The skydiver would fall 6400 feet.